

A characterization of p -bases of rings of constants

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Abstract

We obtain two equivalent conditions for m polynomials in n variables to form a p -basis of a ring of constants of some polynomial K -derivation, where K is a UFD of characteristic $p > 0$. One of these conditions involves jacobians, and the second – some properties of factors. In the case of $m = n$ this extends the known theorem of Nourdin, and we obtain a new formulation of the jacobian conjecture in positive characteristic.

1 Introduction

In this paper we give the full characterization of p -bases of kernels of polynomial derivations (Theorem 4.4). We refer to the sufficient condition (i) and the necessary condition (iv) from [9], Theorem 2.3. Namely, we show that in the case of the polynomial algebra, the condition (i) is also necessary, and we strengthen the condition (iv) to make it also sufficient. The crucial fact we need to realize this aim is the positive characteristic version of Freudenburg's Lemma for m polynomials, where the zero characteristic version was obtained in [5], Theorem 4.1. Note also that the characterization of one-element p -bases was obtained in [6], Theorem 4.2. Let us sketch more precisely all these connections.

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One-element p -bases

Nowicki and Nagata in [17] obtained some interesting results concerning rings of constants (kernels) of k -derivations of the polynomial algebra $k[x, y]$, where k is a field. They showed that such a ring is always (except zero derivation) of the form:

- $k[f]$ for some $f \in k[x, y]$ if $\text{char } k = 0$,
- $k[x^2, y^2, f]$ for some $f \in k[x, y]$ if $\text{char } k = 2$.

In the case of $\text{char } k = p > 2$ they gave an example of a nonzero derivation, which ring of constants is not of the form $k[x^p, y^p, f]$.

The present author has been studying the rings of constants of derivations in positive characteristic, especially the properties of such single generators (that is, one-element p -bases over the respective subring). Hence, we ask, when $k[x_1^p, \dots, x_n^p, f]$ is the ring of constants of a k -derivation, where $\text{char } k = p > 0$ and $f \in k[x_1, \dots, x_n] \setminus k[x_1^p, \dots, x_n^p]$. It was proven in [11] that if f is homogeneous modulo p of a nonzero degree, the following conditions are both necessary and sufficient:

- (1) $\gcd\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = 1$,
- (2) f has no square factors and no factors from $k[x_1^p, \dots, x_n^p] \setminus k$.

In [6] the author discussed sufficient conditions and necessary conditions on various levels of generality. In particular, the condition (1) appeared to be sufficient for arbitrary polynomial f . The final characterization was obtained in [10]. It was shown that the above condition (1) is, in general, also necessary. The other equivalent condition has been obtained in the following form:

- (3) for every $b, c \in k[x_1^p, \dots, x_n^p]$ such that $\gcd(b, c) = 1$ the polynomial $bf + c$ has no square factors and no factors from $k[x_1^p, \dots, x_n^p] \setminus k$.

Many-element p -bases

Nousiainen in [15] (see [13] or [14]) proved that, given polynomials $f_1, \dots, f_n \in k[x_1, \dots, x_n]$, where k is a field of characteristic $p > 0$, then the jacobian condition $\det \left[\frac{\partial f_i}{\partial x_j} \right] \in k \setminus \{0\}$ holds if and only if f_1, \dots, f_n form a p -basis of $k[x_1, \dots, x_n]$ over $k[x_1^p, \dots, x_n^p]$, that is, $k[x_1, \dots, x_n] = k[x_1^p, \dots, x_n^p, f_1, \dots, f_n]$. Note also that the conditions for existence of p -bases of ring extensions have been recently studied by Ono ([18], [19]).

The jacobian condition is an analog of the above condition (1) for n polynomials. Then it is natural to ask, for arbitrary $m \in \{1, \dots, n\}$, when m polynomials form a p -basis of a ring of constants of a k -derivation ($k[x_1, \dots, x_n]$ is the only such a ring if $m = n$).

Consider the following conditions for polynomials $f_1, \dots, f_m \in K[x_1, \dots, x_n]$, where K is a UFD of characteristic $p > 0$, and $m > 1$:

$$(i) \quad \gcd \left(\begin{vmatrix} \frac{\partial f_1}{\partial x_{j_1}} & \dots & \frac{\partial f_1}{\partial x_{j_m}} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_{j_1}} & \dots & \frac{\partial f_m}{\partial x_{j_m}} \end{vmatrix}; 1 \leq j_1 < \dots < j_m \leq n \right) = 1,$$

(ii) *the polynomials f_1, \dots, f_m form a p -basis of a ring of constants of some K -derivation,*

(iii) *the polynomials f_1, \dots, f_m are p -independent and*

$$\gcd \left(\begin{vmatrix} \frac{\partial f_{i_1}}{\partial x_{j_1}} & \frac{\partial f_{i_1}}{\partial x_{j_2}} \\ \frac{\partial f_{i_2}}{\partial x_{j_1}} & \frac{\partial f_{i_2}}{\partial x_{j_2}} \end{vmatrix}; 1 \leq j_1 < j_2 \leq n \right) = 1 \text{ for every } i_1 < i_2,$$

(iv) *the polynomials f_1, \dots, f_m are p -independent and, for every $h_1, \dots, h_m \in K[x_1^p, \dots, x_n^p]$, the polynomials $f_1 + h_1, \dots, f_m + h_m$ are pairwise coprime, have no square factors and no (noninvertible) factors from $K[x_1^p, \dots, x_n^p]$.*

The following implications were obtained in [9], Theorem 2.3:

$$\begin{array}{ccc} (i) & \Rightarrow & (ii) \\ \Downarrow & & \Downarrow \\ (iii) & \Rightarrow & (iv). \end{array}$$

It was clear that the condition (iv) is, in general, not sufficient for (ii), and that the necessity of (i) remains an open question.

The aim of this paper is to prove that the above condition (i) is also necessary for (ii), and to modify the above condition (iv) to make it both necessary and sufficient.

Freudentburg's lemma

The main preparatory result we need to close the chain of implications in Theorem 4.4 is the positive characteristic version of Freudentburg's lemma

for m polynomials. The original version of this lemma was presented by Freudenburg in [4] for one polynomial in two variables over \mathbb{C} .

Lemma 1.1 (Freudenburg). *Given a polynomial $f \in \mathbb{C}[x, y]$, suppose $g \in \mathbb{C}[x, y]$ is an irreducible non-constant divisor of both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Then there exists $c \in \mathbb{C}$ such that g divides $f + c$.*

This fact was generalized to polynomials in n variables over an arbitrary algebraically closed field of characteristic zero by van den Essen, Nowicki and Tyc in [3], Proposition 2.1.

Proposition 1.2 (van den Essen, Nowicki, Tyc). *Let k be an algebraically closed field of characteristic zero. Let P be a prime ideal in $k[x_1, \dots, x_n]$ and $f \in k[x_1, \dots, x_n]$. If for each i the partial derivative $\frac{\partial f}{\partial x_i}$ belongs to P , then there exists $c \in k$ such that $f - c \in P$.*

The present author obtained the following generalization for one polynomial over an arbitrary field (even a UFD).

Theorem 1.3 ([6], Theorem 3.1). *Let K be a UFD, let P be a prime ideal of $K[x_1, \dots, x_n]$. Consider a polynomial $f \in K[x_1, \dots, x_n]$ such that $\frac{\partial f}{\partial x_i} \in P$ for $i = 1, \dots, n$.*

- a) *If $\text{char } K = 0$, then there exists an irreducible polynomial $W(T) \in K[T]$ such that $W(f) \in P$.*
- b) *If $\text{char } K = p > 0$, then there exist $b, c \in K[x_1^p, \dots, x_n^p]$ such that $\gcd(b, c) \sim 1$, $b \notin P$ and $bf + c \in P$.*

The version for n polynomials in n variables over a field of characteristic zero has the following form.

Theorem 1.4 ([5], Theorem 4.1). *Let k be a field of characteristic zero, let $f_1, \dots, f_n \in k[x_1, \dots, x_n]$ be arbitrary polynomials, and let $g \in k[x_1, \dots, x_n]$ be an irreducible polynomial. The following conditions are equivalent:*

- (i) *g divides $\det \left[\frac{\partial f_i}{\partial x_j} \right]$,*
- (ii) *g^2 divides $w(f_1, \dots, f_n)$ for some irreducible polynomial $w \in k[x_1, \dots, x_n]$.*

It is remarkable that as consequence of the above theorem we obtain the characterization of polynomial endomorphisms satisfying the jacobian condition as those mapping irreducible polynomials to square-free polynomials ([5], Theorem 5.1).

In this paper we will obtain Freudenburg's lemma for m polynomials in positive characteristic in two forms: Proposition 3.5 and Theorem 3.6.

The main theorem and the connection with the jacobian conjecture

In Theorem 4.4 we obtain the final characterization of p -bases of rings of constants of polynomial derivations. In the case of n polynomials in n variables it supplements the theorem of Nousiainen with the condition (3) of Theorem 4.4. Hence, following Adjmagbo ([1], see [2], 10.3.16, p. 261), we can reformulate the jacobian conjecture in positive characteristic in the form:

”If polynomials $f_1, \dots, f_n \in \mathbb{F}_p[x_1, \dots, x_n]$ satisfy the condition (3) of Theorem 4.4 (with $K = \mathbb{F}_p$) and p does not divide the degree of the field extension $\mathbb{F}_p(f_1, \dots, f_n) \subset \mathbb{F}_p(x_1, \dots, x_n)$, then $\mathbb{F}_p[f_1, \dots, f_n] = \mathbb{F}_p[x_1, \dots, x_n]$.”

By the theorem of Adjmagbo ([1], see [2], Proposition 10.3.17, p. 261), if the above property holds for all $n \geq 1$ and all primes p , then the jacobian conjecture is true.

2 Rings of constants of derivations and p -bases

Let A be a domain (that is, a commutative ring with unity, without zero divisors) of characteristic $p > 0$. Let B be a subring of A , containing A^p , where $A^p = \{a^p, a \in A\}$. As the main example one may consider the polynomial algebra $A = K[x_1, \dots, x_n]$ and its subalgebra $B = K[x_1^p, \dots, x_n^p]$, where K is a domain of characteristic $p > 0$.

We will use the multi-index notation. Denote:

$$\Omega_m = \{(\alpha_1, \dots, \alpha_m), 0 \leq \alpha_1, \dots, \alpha_m < p\}.$$

Given elements $f_1, \dots, f_m \in A$, $m \geq 1$, and $\alpha = (\alpha_1, \dots, \alpha_m) \in \Omega_m$, we put $f^\alpha = f_1^{\alpha_1} \dots f_m^{\alpha_m}$. If $\alpha = (0, \dots, 0)$, then we put $f^\alpha = 1$.

Recall the definition of a p -basis ([12], p. 269).

Definition 2.1. *The elements $f_1, \dots, f_m \in A$ are called:*

- a) *p -independent over B , if the elements of the form f^α , where $\alpha \in \Omega_m$, are linearly independent over B ,*
- b) *a p -basis of R over B , where R is a subring of A , containing B , if the elements of the form f^α , where $\alpha \in \Omega_m$, form a basis of R as a B -module.*

A single element $f \in A$ is p -independent over B if and only if $f \notin B_0$, where B_0 denotes the field of fractions of B , and in this case the degree of the field extension $B_0 \subset B_0(f)$ equals p . In general, the elements $f_1, \dots, f_m \in A$ are p -independent over B if and only if the degree of the field extension $B_0 \subset B_0(f_1, \dots, f_m)$ equals p^m . The elements $f_1, \dots, f_m \in A$ form a p -basis of R over B if and only if they are p -independent over B and generate R as a B -algebra. Note also that, if the elements $f_1, \dots, f_m \in A$ form a p -basis of R over B , then every element $a \in R$ can be uniquely presented in the form

$$a = \sum_{\alpha \in \Omega_m} b_\alpha f^\alpha,$$

where $b_\alpha \in B$ for $\alpha \in \Omega_m$.

Given elements $f_1, \dots, f_m \in A$, we define the following subring of A :

$$C_B(f_1, \dots, f_m) = B_0(f_1, \dots, f_m) \cap A = B_0[f_1, \dots, f_m] \cap A.$$

If d is a derivation of A , then its kernel is called the ring of constants, and is denoted by A^d (see [16] for a general reference on derivations and rings of constants). Recall ([11], Theorem 1.1 and [8], Theorem 2.5, see also [7], Theorem 3.1) that every ring of constants R of some B -derivation of A satisfies the conditions

$$B \subset R \quad \text{and} \quad R_0 \cap A = R.$$

Observe that $C_B(f_1, \dots, f_m)$ is contained in every ring of constants of a B -derivation, containing the elements f_1, \dots, f_m . If A is finitely generated as a B -algebra, then every subring $R \subset A$ satisfying the above conditions is a ring of constants of some B -derivation of A . Hence, under this assumption, $C_B(f_1, \dots, f_m)$ is the smallest (with respect to inclusion) ring of constants of a B -derivation containing the elements f_1, \dots, f_m (in particular, $B_0 \cap A$ is the smallest ring of constants of a B -derivation). Moreover, in this case, every ring of constants of a B -derivation is of the form $C_B(f_1, \dots, f_m)$ for some elements $f_1, \dots, f_m \in A$, which may be chosen p -independent over B . For details, see [8] and [11]. Note also that if A is a K -algebra, where K is a domain of characteristic $p > 0$, then every K -derivation of A is a B -derivation, where $B = KA^p$ (and then $A^p \subset B$).

We are especially interested in p -bases of rings of constants. This condition may be formulated in several equivalent forms.

Lemma 2.2. *Given arbitrary elements $f_1, \dots, f_m \in A$, consider the following conditions:*

- (1) f_1, \dots, f_m form a p -basis (over B) of the ring of constants of some B -derivation,
- (2) f_1, \dots, f_m are p -independent over B and $B[f_1, \dots, f_m]$ is a ring of constants of some B -derivation,
- (3) f_1, \dots, f_m are p -independent over B and $C_B(f_1, \dots, f_m) = B[f_1, \dots, f_m]$,
- (4) f_1, \dots, f_m form a p -basis of $C_B(f_1, \dots, f_m)$ over B .

The following implications hold:

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4).$$

Moreover, if A is finitely generated as a B -algebra, then all the conditions (1) – (4) are equivalent.

Proof. (1) \Rightarrow (2) If f_1, \dots, f_m form a p -basis of a ring R , then $R = B[f_1, \dots, f_m]$.

(2) \Rightarrow (1) If f_1, \dots, f_m are p -independent over B , then f_1, \dots, f_m form a p -basis of $B[f_1, \dots, f_m]$.

(2) \Rightarrow (3) Consider the ring $R = B[f_1, \dots, f_m]$. We have $R \subset C_B(f_1, \dots, f_m)$. On the other hand, $C_B(f_1, \dots, f_m)$ is contained in every ring of constants of a B -derivation, containing the elements f_1, \dots, f_m , so if R is a ring of constants of some B -derivation, then $R = C_B(f_1, \dots, f_m)$.

(3) \Leftrightarrow (4) This equivalence follows directly from the definition of a p -basis.

If A is finitely generated as a B -algebra, the implication (3) \Rightarrow (2) follows from the fact that $C_B(f_1, \dots, f_m)$ is a ring of constants of a B -derivation. \square

The following technical lemma will be useful in the proof of Lemma 4.1. It is a generalization of Lemma 1.3 from [10].

Lemma 2.3. Assume that $B_0 \cap A = B$. Let $f_1, \dots, f_m \in A$ be p -independent over B . Then the following conditions are equivalent:

- (1) $C_B(f_1, \dots, f_m) = B[f_1, \dots, f_m]$,
- (2) for every $b \in B \setminus \{0\}$ and $(a_\alpha \in B, \alpha \in \Omega_m)$, if $b \mid \sum_{\alpha \in \Omega_m} a_\alpha f^\alpha$, then $b \mid a_\alpha$ for every $\alpha \in \Omega_m$.

Proof. (1) \Rightarrow (2) Assume that $C_B(f_1, \dots, f_m) = B[f_1, \dots, f_m]$. Consider $b \in B \setminus \{0\}$ and $(a_\alpha \in B, \alpha \in \Omega_m)$ such that $b \mid \sum_{\alpha \in \Omega_m} a_\alpha f^\alpha$.

The element $w = \sum_{\alpha \in \Omega_m} \frac{a_\alpha}{b} f^\alpha$ belongs to $B_0[f_1, \dots, f_m]$ and A , so $w \in C_B(f_1, \dots, f_m)$. By the assumption, $w \in B[f_1, \dots, f_m]$, so $w = \sum_{\alpha \in \Omega_m} c_\alpha f^\alpha$,

where $c_\alpha \in B$ for $\alpha \in \Omega_m$. The elements of the form f^α , where $\alpha \in \Omega_m$, are linearly independent over B , hence also over B_0 . Therefore, the equality

$$\sum_{\alpha \in \Omega_m} \frac{a_\alpha}{b} f^\alpha = \sum_{\alpha \in \Omega_m} c_\alpha f^\alpha$$

yields that, for every $\alpha \in \Omega_m$, $\frac{a_\alpha}{b} = c_\alpha$, that is, $b \mid a_\alpha$.

(2) \Rightarrow (1) Assume that, for every $b \in B$ and $(a_\alpha \in B, \alpha \in \Omega_m)$, if $b \mid \sum_{\alpha \in \Omega_m} a_\alpha f^\alpha$, then $b \mid a_\alpha$ for every $\alpha \in \Omega_m$. Consider arbitrary element $w \in C_B(f_1, \dots, f_m)$, $w = \sum_{\alpha \in \Omega_m} c_\alpha f^\alpha$, where $c_\alpha \in B_0$ for $\alpha \in \Omega_m$. Of course, we can present each c_α as $\frac{a_\alpha}{b}$, where $a_\alpha \in B$ for $\alpha \in \Omega_m$ and b is a common denominator for all $\alpha \in \Omega_m$, $b \in B$. Since $w \in A$, we have $b \mid \sum_{\alpha \in \Omega_m} a_\alpha f^\alpha$. Then, by the assumption, for every $\alpha \in \Omega_m$, $b \mid a_\alpha$, that is, $c_\alpha \in A$, so $c_\alpha \in B$ (because $B_0 \cap A = B$). Finally, $w \in B[f_1, \dots, f_m]$. \square

The next lemma follows directly from the proof of Proposition 3.3 a) in [8].

Lemma 2.4. *If the elements $f_1, \dots, f_m \in A$ form a p -basis of $C_B(f_1, \dots, f_m)$ over B , then $B_0 \cap A = B$.*

3 A generalization of Freudenburg's lemma

Throughout this section K is a UFD of characteristic $p > 0$ and $A = K[x_1, \dots, x_n]$ is the polynomial K -algebra in n variables. We put $B = K[x_1^p, \dots, x_n^p]$. We consider arbitrary polynomials $f_1, \dots, f_m \in A$, where $m \geq 1$. We also denote: $R_i = B[f_1, \dots, \widehat{f_i}, \dots, f_m]$ for $i \in \{1, \dots, m\}$ and $R_{ij} = B[f_1, \dots, \widehat{f_i}, \dots, \widehat{f_j}, \dots, f_m]$ for $i, j \in \{1, \dots, m\}$, $i \neq j$, where $\widehat{f_i}$ means that the element f_i is omitted.

For arbitrary $j_1, \dots, j_m \in \{1, \dots, n\}$ we denote by $\text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m}$ the jacobian determinant of f_1, \dots, f_m with respect to x_{j_1}, \dots, x_{j_m} . By $\text{Jac}(f_1, \dots, f_m)$ we denote the ideal generated by all determinants of the form $\text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m}$, where $j_1, \dots, j_m \in \{1, \dots, n\}$. Moreover, following [9], we introduce the notion of a differential gcd of f_1, \dots, f_m :

$$\text{dgcd}(f_1, \dots, f_m) = \text{gcd} \left(\text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m}, j_1, \dots, j_m \in \{1, \dots, n\} \right).$$

If $\text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m} = 0$ for every $j_1, \dots, j_m \in \{1, \dots, n\}$, then we put $\text{dgcd}(f_1, \dots, f_m) = 0$.

Example 3.1. Consider arbitrary $j_1, \dots, j_m \in \{1, \dots, n\}$ and $i \in \{1, \dots, m\}$. Denote by d_i the K -derivation of A defined by

$$d_i(f) = \text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_{i-1}, f, f_{i+1}, \dots, f_m}$$

for $f \in A$. Observe that $d_i(f_i) = \text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m}$ and $d_i(f_j) = 0$ for $j \neq i$. Hence, $R_i \subset A^{d_i}$.

The statement a) in the following lemma is a consequence of the generalized Laplace Expansion Formula. The statements b) and c) follow directly from a), compare the proof of implication (i) \Rightarrow (iii) in [9], Theorem 2.3.

Lemma 3.2. Consider arbitrary pairwise different $i_1, \dots, i_r \in \{1, \dots, m\}$, where $1 \leq r \leq m$. Then:

- a) $\text{Jac}(f_1, \dots, f_m) \subset \text{Jac}(f_{i_1}, \dots, f_{i_r})$,
- b) $\text{dgcd}(f_{i_1}, \dots, f_{i_r}) \mid \text{dgcd}(f_1, \dots, f_m)$, whenever $\text{dgcd}(f_{i_1}, \dots, f_{i_r}) \neq 0$,
- c) $\text{dgcd}(f_1, \dots, f_m) = 0$ if $\text{dgcd}(f_{i_1}, \dots, f_{i_r}) = 0$.

Let Q be a prime ideal of A . We denote by \overline{A} the factor algebra A/Q . For arbitrary element $a \in A$ we denote by \overline{a} the coset of a in \overline{A} , that is, $\overline{a} = a + Q$. For a subring $T \subset A$ we denote by \overline{T} the canonical homomorphic image of T in \overline{A} .

Lemma 3.3. Consider a subring $T \subset A$ such that $B \subset T$. Given a polynomial $f \in A$, the element \overline{f} is p -dependent over \overline{T} if and only if there exist $b, c \in T$, $b \notin Q$, such that $bf + c \in Q$.

Proof. The element \overline{f} is p -dependent over \overline{T} if and only if it belongs to $(\overline{T})_0$, that is, it can be presented in the form $-\frac{\overline{c}}{\overline{b}}$ for some $b, c \in T$ such that $\overline{b} \neq \overline{0}$, that is, $b \notin Q$. We obtain the equality $\overline{b} \cdot \overline{f} + \overline{c} = \overline{0}$, hence $bf + c \in Q$. \square

The following proposition is a positive characteristic analog (for arbitrary number of polynomials) of Lemma 3.1 from [6].

Proposition 3.4. a) The inclusion $\text{Jac}(f_1, \dots, f_m) \subset Q$ holds if and only if the following condition is satisfied for some $i \in \{1, \dots, m\}$:

(*) there exist $s_1, \dots, s_m \in A$, where $s_i \notin Q$, such that $s_1 d(f_1) + \dots + s_m d(f_m) \in Q$ for every K -derivation d of A .

b) If the above condition (*) is satisfied for a given $i \in \{1, \dots, m\}$, then the element $\overline{f_i}$ is p -dependent over $\overline{R_i}$.

Proof. **a)** The condition $\text{Jac}(f_1, \dots, f_m) \subset Q$ holds if and only if the rank of the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

over the field A_0 is less than m . Repeating the arguments from the proof of Lemma 3.1 a) in [5], we obtain that this is equivalent to (*).

b) Assume that (*) holds for a given i . Consider arbitrary \overline{R}_i -derivation δ of \overline{A} . Repeating the arguments from the proof of Lemma 3.1 b) in [5], we obtain that $\delta(\overline{f}_i) = \overline{0}$. Hence, \overline{f}_i belongs to the smallest ring of constants of a \overline{R}_i -derivation of \overline{A} , that is, $\overline{f}_i \in (\overline{R}_i)_0 \cap \overline{A}$, so \overline{f}_i is p -dependent over \overline{R}_i . \square

Proposition 3.5 and Theorem 3.6 are two forms of the positive characteristic Freudenburg's lemma for m polynomials. The first one follows from Lemma 3.3 and Proposition 3.4. It is a generalization of Theorem 3.1 b) from [6].

Proposition 3.5. *Consider arbitrary polynomials $f_1, \dots, f_m \in K[x_1, \dots, x_n]$, where K is a UFD of characteristic $p > 0$ and $m, n \geq 1$. If $\text{Jac}(f_1, \dots, f_m) \subset Q$, where Q is a prime ideal of A , then there exist $i \in \{1, \dots, m\}$ and*

$$b, c \in K[x_1^p, \dots, x_n^p, f_1, \dots, \widehat{f}_i, \dots, f_m],$$

$b \notin Q$, such that $bf_i + c \in Q$.

The following theorem is a generalization of Proposition 3.3 b) from [6] and a positive characteristic analog of Theorem 4.1 from [5].

Theorem 3.6. *Let $A = K[x_1, \dots, x_n]$ be the polynomial K -algebra, where K is a UFD of characteristic $p > 0$ and $n \geq 1$. Put $B = K[x_1^p, \dots, x_n^p]$. Consider arbitrary polynomials $f_1, \dots, f_m \in A$, where $m \geq 1$, and denote $R_i = B[f_1, \dots, \widehat{f}_i, \dots, f_m]$ and $R_{ij} = B[f_1, \dots, \widehat{f}_i, \dots, \widehat{f}_j, \dots, f_m]$, $i \neq j$.*

Then $\text{dgcd}(f_1, \dots, f_m)$ is divisible by an irreducible polynomial $g \in A$ if and only if at least one of the following conditions hold:

- (i) *$g \notin B$ and $g^2 \mid bf_i + c$ for some $i \in \{1, \dots, m\}$ and $b, c \in R_i$ such that $g \nmid b$,*
- (ii) *$g \in B$ and $g \mid bf_i + c$ for some $i \in \{1, \dots, m\}$ and $b, c \in R_i$ such that $g \nmid b$,*

(iii) $g \mid b_1 f_i + c_1$ and $g \mid b_2 f_j + c_2$ for some $i, j \in \{1, \dots, m\}$, $i \neq j$, and $b_1, b_2, c_1, c_2 \in R_{ij}$ such that $g \nmid b_1$ and $g \nmid b_2$.

Proof. (\Rightarrow) We follow the arguments from the proof of implication (i) \Rightarrow (ii) in [5], Theorem 4.1. Assume that the polynomial $\text{dgcd}(f_1, \dots, f_m)$ is divisible by an irreducible polynomial $g \in A$. Then $\text{Jac}(f_1, \dots, f_m) \subset (g)$, where (g) denotes the principal ideal generated by g , so, by Proposition 3.5, there exist $i \in \{1, \dots, m\}$, $b, c \in R_i$ such that $g \nmid b$ and $g \mid b f_i + c$, that is,

$$(1) \quad b f_i + c = g h$$

for some $h \in A$. If $g \in B$, the condition (ii) holds. If $g \notin B$ and $g \mid h$, the condition (i) holds. Now assume that $g \notin B$ and $g \nmid h$.

Consider arbitrary $j_1, \dots, j_m \in \{1, \dots, n\}$ and the derivation d_i from Example 3.1. Note that $d_i(b) = 0$ and $d_i(c) = 0$, because $b, c \in R_i$. Applying the derivation d_i to both sides of (1) we obtain

$$b \text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m} = g d_i(h) + d_i(g) h.$$

By the assumption, $g \mid \text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m}$, so $g \mid d_i(g)$.

Hence, the determinant

$$\text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_{i-1}, g, f_{i+1}, \dots, f_m} = d_i(g)$$

is divisible by g for every $j_1, \dots, j_m \in \{1, \dots, n\}$, so, by Proposition 3.4 a), there exist $s_1, \dots, s_m \in A$, where $g \nmid s_j$ for some $j \in \{1, \dots, m\}$, such that

$$g \mid s_1 d(f_1) + \dots + s_i d(g) + \dots + s_m d(f_m)$$

for every K -derivation d of A . Note that the polynomials s_j , $j \neq i$, can not all together be divisible by g . Indeed, in this case we would have $g \nmid s_i$ and $g \mid s_i d(g)$, so $g \mid d(g)$ for every K -derivation d , what is not true for $d = \frac{\partial}{\partial x_i}$ such that $\frac{\partial g}{\partial x_i} \neq 0$ (recall that $g \notin B$). Thus $g \nmid s_j$ for some $j \neq i$, so, by Proposition 3.4 b), $\overline{f_j}$ is p -dependent over $\overline{B}[\overline{f_1}, \dots, \overline{f_{i-1}}, \overline{g}, \overline{f_{i+1}}, \dots, \widehat{\overline{f_j}}, \dots, \overline{f_m}] = \overline{R_{ij}}$.

Since $\overline{f_i}$ is p -dependent over $\overline{R_i} = \overline{R_{ij}[\overline{f_j}]}$, we obtain that both $\overline{f_i}$ and $\overline{f_j}$ are p -dependent over $\overline{R_{ij}}$. Then, by Lemma 3.3, $g \mid b_1 f_i + c_1$ and $g \mid b_2 f_j + c_2$ for some $b_1, b_2, c_1, c_2 \in R_{ij}$ such that $g \nmid b_1$ and $g \nmid b_2$, and the condition (iii) holds.

(\Leftarrow) Assume that the condition (i) holds, that is, $g \notin B$ and $g^2 \mid b f_i + c$ for some $i \in \{1, \dots, m\}$, $b, c \in R_i$ such that $g \nmid b$. Then $b f_i + c = g^2 h$

for some $h \in A$. Applying the derivation d_i from Example 3.1 for arbitrary $j_1, \dots, j_m \in \{1, \dots, n\}$ we obtain that $b \text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m} = 2gd_i(g)h + g^2d_i(h)$, so $g \mid \text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m}$. Hence, $g \mid \text{dgc}d(f_1, \dots, f_m)$.

Analogously, if the condition (ii) holds, then $g \mid \text{dgc}d(f_1, \dots, f_m)$ as well.

Finally, assume that the condition (iii) holds, that is, $g \mid b_1f_i + c_1$ and $g \mid b_2f_j + c_2$ for some $i, j \in \{1, \dots, m\}$, $i \neq j$, $b_1, b_2, c_1, c_2 \in R_{ij}$ such that $g \nmid b_1$ and $g \nmid b_2$. It is easy to check that if $g \mid h_1$ and $g \mid h_2$, where $h_1, h_2 \in A$, then $g \mid \text{jac}_{l_1, l_2}^{h_1, h_2}$ for every l_1, l_2 , so $g \mid \text{dgc}d(h_1, h_2)$. Hence,

$$g \mid \text{dgc}d(b_1f_i + c_1, b_2f_j + c_2),$$

and then, by Lemma 3.2 b), c),

$$g \mid \text{dgc}d(f_1, \dots, b_1f_i + c_1, \dots, b_2f_j + c_2, \dots, f_m).$$

On the other side, using the arguments from Example 3.1, for arbitrary $j_1, \dots, j_m \in \{1, \dots, n\}$ we have

$$\text{jac}_{j_1, \dots, j_m}^{f_1, \dots, b_1f_i + c_1, \dots, b_2f_j + c_2, \dots, f_m} = b_1 \text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_i, \dots, b_2f_j + c_2, \dots, f_m} = b_1b_2 \text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_i, \dots, f_j, \dots, f_m}.$$

Since $g \nmid b_1b_2$, we obtain that $g \mid \text{jac}_{j_1, \dots, j_m}^{f_1, \dots, f_m}$, so, finally, $g \mid \text{dgc}d(f_1, \dots, f_m)$. \square

4 A characterization of p -bases of rings of constants

Recall that if A is a domain, then two elements $a, b \in A$ are called associated, and we denote it $a \sim b$, if $a = bc$ for some invertible element $c \in A$. An element $a \in A$ is called square-free if it is not divisible by a square of any noninvertible element. If $\text{char } A = p > 0$ and B is a subring containing A^p , then an element $a \in A$ is called B -free if it is not divisible by any noninvertible element of B . (Note that if $A^p \subset B$, then an element of B is invertible in A if and only if it is invertible in B).

The following lemma contains some necessary conditions for one and two elements of a UFD to form a p -basis of the respective ring. The case of one element was obtained in [6], Theorem 4.2, for a polynomial algebra, but the arguments remain valid for arbitrary UFD.

Lemma 4.1. *Let A be UFD of characteristic $p > 0$, let B be a subring of A , containing A^p .*

a) *If an element $f \in A$ forms a p -basis of $C_B(f)$ over B , then $bf + c$ is square-free and B -free for every $b, c \in B$ such that $b \neq 0$ and $\gcd(b, c) \sim 1$.*

b) *If elements $f_1, f_2 \in A$ form a p -basis of $C_B(f_1, f_2)$ over B , then $\gcd(b_1f_1 + c_1, b_2f_2 + c_2) \sim 1$ for every $b_1, b_2, c_1, c_2 \in B$ such that $b_i \neq 0$ and $\gcd(b_i, c_i) \sim 1$ for $i = 1, 2$.*

Proof. **a)** Assume that f forms a p -basis of $C_B(f)$ over B . Consider arbitrary elements $b, c \in B$ such that $b \neq 0$ and $\gcd(b, c) \sim 1$. If $h \mid bf + c$ for some noninvertible element $h \in B$, then, by Lemmas 2.3 and 2.4, $h \mid b$ and $h \mid c$, a contradiction.

Now, suppose that $g^2 \mid bf + c$ for some irreducible element $g \in A$. Note that $g^p \mid g^{2(p-1)}$ and $g^{2(p-1)} \mid (bf + c)^{p-1}$, so $g^p \mid (bf + c)^{p-1}$. We have $(bf + c)^{p-1} = b^{p-1}f^{p-1} + \dots + c^{p-1}$, and by Lemmas 2.3 and 2.4 we obtain that $g^p \mid b^{p-1}$ and $g^p \mid c^{p-1}$, so $g \mid b$ and $g \mid c$, a contradiction.

b) Assume that $f_1, f_2 \in A$ form a p -basis of $C_B(f_1, f_2)$ over B . Suppose that $g \mid b_1f_1 + c_1$ and $g \mid b_2f_2 + c_2$ for some irreducible element $g \in A$ and some $b_1, b_2, c_1, c_2 \in B$ such that $b_i \neq 0$ and $\gcd(b_i, c_i) \sim 1$ for $i = 1, 2$. Then $g^p \mid (b_1f_1 + c_1)^{p-1}(b_2f_2 + c_2)$, that is,

$$g^p \mid b_1^{p-1}b_2f_1^{p-1}f_2 + b_1^{p-1}c_2f_1^{p-1} + \dots + c_1^{p-1}b_2f_2 + c_1^{p-1}c_2.$$

By Lemmas 2.3 and 2.4 we obtain that $g^p \mid b_1^{p-1}b_2$, so $g \mid b_1$ or $g \mid b_2$. And this is impossible: if $g \mid b_i$, where $i \in \{1, 2\}$, then $g \mid c_i$, because $g \mid b_if_i + c_i$. \square

In the following proposition we obtain a necessary condition for a p -basis, stronger than (iv) in [9], Theorem 2.3. It will follow from Theorem 4.4, that in the case of the polynomial algebra this condition is also sufficient.

Proposition 4.2. *Let A be UFD of characteristic $p > 0$, let B be a subring of A , containing A^p . Assume that the elements $f_1, \dots, f_m \in A$ form a p -basis of $C_B(f_1, \dots, f_m)$ over B . Denote: $R_i = B[f_1, \dots, \widehat{f_i}, \dots, f_m]$, $R_{ij} = B[f_1, \dots, \widehat{f_i}, \dots, \widehat{f_j}, \dots, f_m]$. Then the following conditions hold:*

(i) *the element $bf_i + c$ is square-free and B -free for every $i \in \{1, \dots, m\}$ and $b, c \in R_i$ such that $b \neq 0$ and $\gcd(b, c) \sim 1$,*

(ii) *if $m > 1$, then $\gcd(b_1f_i + c_1, b_2f_j + c_2) \sim 1$ for every $i, j \in \{1, \dots, m\}$, $i \neq j$, and $b_1, b_2, c_1, c_2 \in R_{ij}$ such that $b_1, b_2 \neq 0$, $\gcd(b_1, c_1) \sim 1$, $\gcd(b_2, c_2) \sim 1$.*

Proof. Put $R = C_B(f_1, \dots, f_m)$. Then $R = B[f_1, \dots, f_m]$ by Lemma 2.2. By the definition of a p -basis we see that, for every $i \in \{1, \dots, m\}$, the element f_i forms a p -basis of R over R_i . On the other hand, $C_{R_i}(f_i) = R$, so (i) follows from Lemma 4.1 a), note that R_i -free element is B -free. Similarly, for every $i, j \in \{1, \dots, m\}$, $i \neq j$, the elements f_i, f_j form a p -basis of R over R_{ij} , and $C_{R_{ij}}(f_i, f_j) = R$, so (ii) follows from Lemma 4.1 b). \square

The next lemma will help us apply the conclusions of Theorem 3.6 in the proof of Theorem 4.4.

Lemma 4.3. *Let A be a UFD of characteristic $p > 0$, let B be a subring of A such that $A^p \subset B$. Consider arbitrary element $f \in A$. Assume that $g^\varepsilon \mid bf + c$ for some irreducible element $g \in A$, $\varepsilon \in \{1, 2\}$ and $b, c \in B$ such that $g \nmid b$. Then there exists an element $c' \in B$ such that $g^\varepsilon \mid bf + c'$ and $\gcd(b, c') \sim 1$.*

Proof. Consider the decompositions of b and c into irreducible factors: $b = uq_1^{\alpha_1} \dots q_s^{\alpha_s}$, $c = vr_1^{\beta_1} \dots r_t^{\beta_t}$, where the elements $u, v \in A$ are invertible, $s, t \geq 0$, $q_i \not\sim q_j$ for $i \neq j$, $r_i \not\sim r_j$ for $i \neq j$, $\alpha_i > 0$, $\beta_i > 0$. We may assume that there exists $l \geq 0$, $l \leq s$, such that $q_i \sim r_i$ if $i \leq l$ and $q_i \not\sim r_j$ if $i, j > l$.

Put $h = g^p q_{l+1}^p \dots q_s^p$ ($h = g^p$ if $l = s$) and $c' = c + h$. We have $h \in B$ and $g^\varepsilon \mid h$, so $c' \in B$ and $g^\varepsilon \mid bf + c'$. Note that $q_i \not\sim g$ for each i , because $g \nmid b$. Now, if $i \leq l$, then $q_i \mid c$ and $q_i \nmid h$, so $q_i \nmid c'$. If $i > l$, $i \leq s$, then $q_i \nmid c$ and $q_i \mid h$, so $q_i \nmid c'$. Hence $\gcd(b, c') \sim 1$. \square

Now we can prove the main result of the paper – a characterization of p -bases of rings of constants of polynomial derivations.

Theorem 4.4. *Let K be a UFD of characteristic $p > 0$, consider arbitrary polynomials $f_1, \dots, f_m \in K[x_1, \dots, x_n]$, where $m \in \{1, \dots, n\}$. Denote: $B = K[x_1^p, \dots, x_n^p]$, $R_i = B[f_1, \dots, \widehat{f_i}, \dots, f_m]$, $R_{ij} = B[f_1, \dots, \widehat{f_i}, \dots, \widehat{f_j}, \dots, f_m]$, $i \neq j$.*

The following conditions are equivalent:

- (1) $\text{dgc}(f_1, \dots, f_m) \sim 1$,
- (2) *the polynomials f_1, \dots, f_m form a p -basis of a ring of constants of some K -derivation,*
- (3) *the polynomial $bf_i + c$ is square-free and B -free for every $i \in \{1, \dots, m\}$ and $b, c \in R_i$ such that $b \neq 0$ and $\gcd(b, c) \sim 1$, and, if $m > 1$, then $\gcd(b_1 f_i + c_1, b_2 f_j + c_2) \sim 1$ for every $i, j \in \{1, \dots, m\}$, $i \neq j$, and $b_1, b_2, c_1, c_2 \in R_{ij}$ such that $b_1, b_2 \neq 0$, $\gcd(b_1, c_1) \sim 1$ and $\gcd(b_2, c_2) \sim 1$.*

Proof. (1) \Rightarrow (2) This implication was established in [10], Theorem 2.3 for $m = 1$ and in [9], Theorem 2.3 for $m > 1$.

(2) \Rightarrow (3) This implication follows from Lemma 2.2 and Proposition 4.2.

$\neg(1) \Rightarrow \neg(3)$ Assume that $\text{dgcd}(f_1, \dots, f_m) \not\sim 1$, so $\text{dgcd}(f_1, \dots, f_m)$ is divisible by some irreducible polynomial $g \in A$. Then, by Theorem 3.6, at least one of the conditions (i) – (iii) hold. In each case, using Lemma 4.3, we deduce $\neg(3)$. \square

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